Anti-Derivative (U-Sub) WS 2

1) $\int 3(\sin x)^{-2}\,dx = 3\int \frac{1}{(\sin x)^2}\,dx = 3\int \csc^2 x\,dx$

$= \boxed{-3\cot x + C}$

2) $\int \frac{\ln^6 x}{x}\,dx = \int \frac{u^6}{\cancel{x}}(\cancel{x}\,du) = \int u^6\,du = \frac{u^7}{7} + C$

$= \boxed{\frac{\ln^7|x|}{7} + C}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

3) $\int \frac{dx}{x\ln x} = \int \frac{\cancel{x}\,du}{\cancel{x}(u)} = \int \frac{1}{u}\,du = \ln|u| + C$

$\boxed{= \ln\left|\ln|x|\right| + C}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

4) $\int \sec^4 x\,dx$ ← you want to see 2 different trig functions when you take the antiderivative

Sec x is usually paired with tan x

$= \int \sec^2 x \cdot \sec^2 x\,dx$ use $\tan^2 x + 1 = \sec^2 x$

$= \int \sec^2 x(\tan^2 x + 1)\,dx = \int \cancel{\sec^2 x}(u^2+1)\left(\frac{du}{\cancel{\sec^2 x}}\right)$

$u = \tan x$

$du = \sec^2 x\,dx$

$dx = \frac{du}{\sec^2 x}$

$= \int u^2 + 1\,du = \frac{u^3}{3} + u + C$

$= \boxed{\frac{\tan^3 x}{3} + \tan x + C}$

5) $\int t^{1/3}\cos(t^{4/3} - 8)\,dt = \int \cancel{t^{1/3}}\cos u\left(\frac{3\,du}{4\cancel{t^{1/3}}}\right) = \frac{3}{4}\int \cos u\,du$

$u = t^{4/3} - 8$

$du = \frac{4}{3}t^{1/3}\,dt$

$dt = \frac{3\,du}{4t^{1/3}}$

$= \frac{3}{4}\sin u + C$

$= \boxed{\frac{3}{4}\sin(t^{4/3} - 8) + C}$

6) $\int \frac{dx}{\sin^2 3x} = \int \csc^2 3x \, dx = \int \csc^2 u \left(\frac{du}{3}\right)$

$u = 3x$
$du = 3\,dx$
$dx = \frac{du}{3}$

$= \frac{1}{3}\int \csc^2 u \, du$

$= -\frac{1}{3}\cot u + C$

$\boxed{= -\frac{1}{3}\cot 3x + C}$

7) $\int x \sin^3(5x^2)\,dx = \int x \sin(5x^2) \cdot \sin^2(5x^2)\,dx$

$= \int x \sin(5x^2)\left[1 - \cos^2(5x^2)\right] dx$

$u = \cos(5x^2)$
$du = -\sin(5x^2)(10x)\,dx$
$dx = \frac{du}{-10x\sin(5x^2)}$

$= \int \cancel{x \sin(5x^2)}\,(1-u^2)\left(\frac{du}{\cancel{-10x\sin(5x^2)}}\right)$

$= -\frac{1}{10}\int 1 - u^2 \, du$

$= -\frac{1}{10}\left(u - \frac{u^3}{3}\right) + C$

$= \boxed{\frac{\cos^3(5x^2)}{30} - \frac{\cos(5x^2)}{10} + C}$

8) $f(x) = \int x^3 - 7x^2 + 3x - 8 = \frac{x^4}{4} - \frac{7x^3}{3} + \frac{3x^2}{2} - 8x + C$

$f(2) = -10 \Leftarrow$ initial conditions $\Rightarrow$ solve for C

$-10 = \frac{2^4}{4} - \frac{7(2)^3}{3} + \frac{3(2)^2}{2} - 8(2) + C$

$-10 = 4 - \frac{56}{3} + 6 - 16 + C$

$-10 = -\frac{74}{3} + C$

$C = \frac{44}{3}$

$\boxed{\therefore f(x) = \frac{x^4}{4} - \frac{7x^3}{3} + \frac{3x^2}{2} - 8x + \frac{44}{3}}$

9) $f(x) = \int \sin x \cos^2 x \, dx = \int \cancel{\sin x}\, u^2 \left(\frac{du}{-\cancel{\sin x}}\right)$

$u = \cos x$

$du = -\sin x \, dx$

$dx = \frac{du}{-\sin x}$

$= -\int u^2 \, du$

$= -\frac{u^3}{3} + C$

$= -\frac{\cos^3 x}{3} + C$

$f(\pi) = 1$

initial conditions

$1 = -\frac{\cos^3(\pi)}{3} + C$

$1 = \frac{1}{3} + C$

$C = \frac{2}{3}$

$\therefore \boxed{f(x) = -\frac{\cos^3 x}{3} + \frac{2}{3}}$

10) $f(x) = \int \frac{e^{3x}\,dx}{\sqrt{1-e^{6x}}} = \int \frac{\cancel{e^{3x}}}{\sqrt{1-u^2}} \left(\frac{du}{3\cancel{e^{3x}}}\right) = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$

$e^{6x} = (e^{3x})^2$

$u = e^{3x}$

$du = e^{3x} \cdot 3\,dx$

$dx = \frac{du}{3e^{3x}}$

$= \frac{1}{3}\sin^{-1} u + C$

$= \frac{1}{3}\sin^{-1}(e^{3x}) + C$

$f(0) = \pi$

initial condition

$\pi = \frac{1}{3}\sin^{-1}(e^0) + C$

$\pi = \frac{1}{3}\sin^{-1}(1) + C$

$\pi = \frac{1}{3}\left(\frac{\pi}{2}\right) + C$

$\pi = \frac{\pi}{6} + C$

$C = \frac{5\pi}{6}$

$\therefore \boxed{f(x) = \frac{1}{3}\sin^{-1}(e^{3x}) + \frac{5\pi}{6}}$

11) $f(x) = 5^{3x}$ $\quad f'(x) = 5^{3x} \cdot \ln 5\,(3) = \boxed{3\ln 5\,(5^{3x})}$ or $\boxed{\ln 125\,(5^{3x})}$

12) $$\lim_{x\to 0} 4 \frac{\sin x \cos x - \sin x}{x^2} = 4\left[\frac{(0)(1)-0}{0^2}\right] = \frac{0}{0}$$

indeterminate form use L'Hopital's Rule

$$\lim_{x\to 0} 4 \frac{\cos x \cos x + (-\sin x)(\sin x) - \cos x}{2x}$$

$$\lim_{x\to 0} 4 \frac{\cos^2 x - \sin^2 x - \cos x}{2x} = 4\left[\frac{\cos^2(0) - \sin^2(0) - \cos(0)}{2(0)}\right] = \frac{0}{0}$$

use L'Hopital's again

$$\lim_{x\to 0} 4\left[\frac{-2\cos x \sin x - 2\sin x \cos x + \sin x}{2}\right] = 4\left[\frac{-2(1)(0) - 2(0)(1) + 0}{2}\right]$$

$$\boxed{=0}$$

13) $$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{6}+h\right) - \tan\left(\frac{\pi}{6}\right)}{h}$$ } limit definition of a derivative

$$= \frac{d}{dx}\Big|_{x=\frac{\pi}{6}} \tan x = \sec^2\left(\frac{\pi}{6}\right) = \frac{1}{\cos^2\left(\frac{\pi}{6}\right)} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} = \boxed{\frac{4}{3}}$$

14) $f(x)$ is differentiable $\longrightarrow$ $f(x)$ is continuous

if $f(x)$ is continuous then there should be <u>no</u> jumps or breaks.

$$f(x) = \begin{cases} ax^3 - 6x, & x \le 1 \\ bx^2 + 4, & x > 1 \end{cases}$$ has to be continuous at $x = 1$

top: $a(1)^3 - 6(1) = a - 6$
bottom: $b(1)^2 + 4 = b + 4$
} have to = each other to be continuous.

* $a - 6 = b + 4 \rightarrow$ 1 equation, 2 variables (<u>can't</u> solve)

if a function is differentiable then the derivative exists

$$f'(x) = \begin{cases} 3ax^2 - 6 & , x \le 1 \\ 2bx & , x > 1 \end{cases}$$

for the derivative to exist at 1 the $\lim_{x \to 1} f'(x)$ would have to exist.

top: $3a(1)^2 - 6 = 3a - 6$ have to = each other

bottom: $2b(1) = 2b$

$3a - 6 = 2b$ 2 equations 2 variables (solvable)

* $a - 6 = b + 4$

substitution method:

$a - 6 = b + 4 \Rightarrow b = a - 10$ plug into other equation

$3a - 6 = 2(a - 10)$

$3a - 6 = 2a - 20$

$\boxed{a = -14}$

15) position: $y_1 = \cos 2t$ $y_2 = 4\sin t$

velocity: $v(t) = -2\sin 2t$ $v(t) = 4\cos t$

acceleration: $a(t) = -4\cos 2t$ $a(t) = -4\sin t$

$$-4\cos 2t = -4\sin t$$

The easiest way to solve this equation is to graph both functions

$Y_1: y_1 = -4\cos 2X$

$Y_2: y_2 = -4\sin X$

and count the number of times the 2 graphs intersect for $0 < x < 6$.

Answer: $\boxed{3}$ intersections