Review WS 7

2008 #3

a) $r = 100\text{cm}$ $h = 0.5\text{cm}$ $\frac{dr}{dt} = 2.5 \text{ cm/min}$ $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$

$V = \pi r^2 h$

Even though the shape of the oil spill is a cylinder, the Volume, height and radius are all changing with time.

$$\frac{dV}{dt} = \pi\left((2r)\left(\frac{dr}{dt}\right)(h) + \left(\frac{dh}{dt}\right)(r^2)\right)$$

$$2000 = \pi\left((2\cdot 100)(2.5)(0.5) + \frac{dh}{dt}(100)^2\right)$$

$$2000 = \pi\left(250 + 10000\frac{dh}{dt}\right) = 250\pi + 10000\pi\frac{dh}{dt}$$

$$\Rightarrow \quad 10000\pi\frac{dh}{dt} = 2000 - 250\pi$$

$$\frac{dh}{dt} = \frac{2000 - 250\pi}{10000\pi} = .039 \text{ cm/min}$$

b) $V(t) = 2000t - \int_0^t R(t)\,dt$

$V'(t) = 2000 - R(t) = 0 \Rightarrow R(t) = 2000$

$400\sqrt{t} = 2000$

$\sqrt{t} = 5$

$t = 25$

1	25	36
$+$		$-$

$V'(1) = 2000 - R(1) = 2000 - 400\sqrt{1} = +$

$V'(36) = 2000 - R(36) = 2000 - 400\sqrt{36} = -$

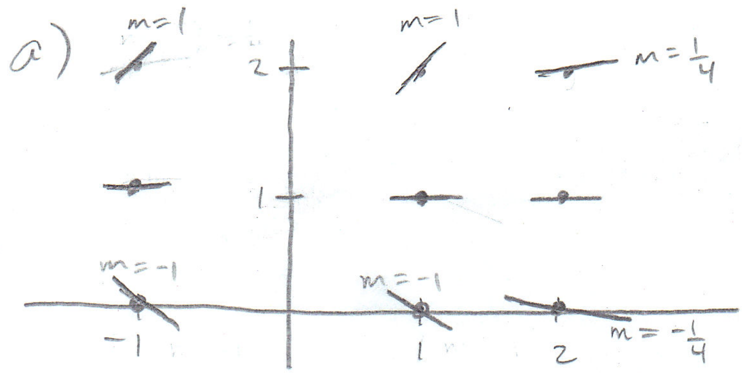
$\therefore$ Volume reaches its maximum volume @ $t = 25$ min

c) $V(t) = 60{,}000 + 2000t - \int_0^t R(t)\,dt$

Volume @ $t = 25$ min : $V(25) = 60{,}000 + 2000(25) - \int_0^{25} R(t)\,dt$

2008 #5

a)



b) $f(2) = 0$

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx$$

$$\ln|y-1| = -x^{-1} + C$$

$$\ln|0-1| = -\frac{1}{2} + C$$

$$\ln 1 = -\frac{1}{2} + C$$

$$C = \frac{1}{2}$$

$$\ln|y-1| = -x^{-1} + \frac{1}{2}$$

$$e^{\ln|y-1|} = e^{-\frac{1}{x}+\frac{1}{2}} = e^{-\frac{1}{x}} \cdot e^{1/2}$$

Because this is a particular solution for $f(2) = 0$

$$|y-1| = -(y-1) = 1-y$$

Now when you plug 0 into $1-y$ it gives a positive number

$$e^{\ln|y-1|} = e^{\ln(1-y)} = e^{-\frac{1}{x}} \cdot e^{\frac{1}{2}}$$

$$\Rightarrow 1-y = e^{-\frac{1}{x}} \cdot e^{\frac{1}{2}}$$

$$\Rightarrow -y = e^{-\frac{1}{x}} \cdot e^{\frac{1}{2}} - 1$$

$$\Rightarrow \boxed{y = 1 - e^{-\frac{1}{x}} \cdot e^{\frac{1}{2}}}$$

c) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 - e^{-\frac{1}{x}} \cdot e^{\frac{1}{2}} = \lim_{x\to\infty} 1 - \left(\frac{e^{\frac{1}{2}}}{e^{\frac{1}{x}}}\right)$

$\Rightarrow \; = 1 - \left(\frac{e^{\frac{1}{2}}}{e^{\frac{1}{\infty}}}\right) = 1 - \left(\frac{e^{\frac{1}{2}}}{e^{0}}\right) = 1 - \frac{e^{\frac{1}{2}}}{1} = 1 - e^{\frac{1}{2}}$

2008 #6 $\quad f(x) = \frac{\ln x}{x}$, $\quad f'(x) = \frac{1 - \ln x}{x^2}$

a) $x = e^2$

$y = f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$

point $\left(e^2, \frac{2}{e^2}\right)$

$m_{tan} = f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{1-2}{e^4} = -\frac{1}{e^4}$

$y - y_1 = m(x - x_1)$

$y - \frac{2}{e^2} = -\frac{1}{e^4}(x - e^2)$

$y = \frac{2}{e^2} - \frac{1}{e^4}(x - e^2)$

b) critical points ($f'(x) = 0$ or $f'(x)$ DNE)

$f'(x)$ DNE @ $x = 0$ but $x > 0$ so it does not count

$f'(x) = 0 \Rightarrow \frac{1 - \ln x}{x^2} = 0$

$1 - \ln x = 0$

$e^{\ln x} = e^1$

$x = e$

1	e	5
+		−

$\therefore$ $f(x)$ has a relative max @ $x = e$ b/c $f'(x)$ changes from + to −.

c) $f''(x) = \dfrac{-\frac{1}{x}(x^2) - (2x)(1-\ln x)}{(x^2)^2} = \dfrac{-x - 2x + 2x\ln x}{x^4}$

$f''(x) = \dfrac{-3x + 2x\ln x}{x^4} = 0$

$-3x + 2x\ln x = 0$

$x(-3 + 2\ln x) = 0$

$x = 0$ can't happen b/c $x > 0$

$-3 + 2\ln x = 0$

$2\ln x = 3$

$\ln x = \frac{3}{2}$

$e^{\ln x} = e^{\frac{3}{2}}$

$x = e^{3/2}$

plug into 2nd derivative

1	$e^{3/2}$	10
$-$		$+$

$\therefore$ the x-value for the point of inflection is $x = e^{3/2}$ b/c $f''(x)$ changes sign.

d) $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{x} = \lim_{x\to 0^+} \frac{1}{x} \cdot \ln x = \frac{1}{0^+} \cdot \ln(0^+) = +\infty \cdot -\infty$

$= -\infty$