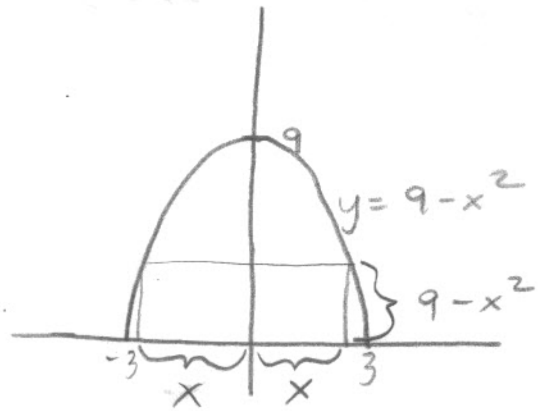
Review WS 3

1) $h(x) = 9 - x^2$



length of rectangle $= x + x = 2x$

width of rectangle $= 9 - x^2$

Area of rectangle $= l \cdot w = 2x(9-x^2)$

$A = 18x - 2x^3$

Maximize Area $\rightarrow$ take 1st der

$$\frac{dA}{dx} = 18 - 6x^2$$

Find critical points ($\frac{dA}{dx} = 0$, $\frac{dA}{dx}$ undef)

(undef): $18-6x^2$ is a polynomial ie globally continuous.
$\therefore$ $18-6x^2$ is never undefined.

$(=0)$: $18 - 6x^2 = 0$
$6x^2 = 18$
$x^2 = 3$
$x = \pm\sqrt{3}$ $\rightarrow$ sign chart

-5	$-\sqrt{3}$	0	$\sqrt{3}$	5
$-$		$+$		$-$

$\therefore$ maximum @ $x = \sqrt{3}$ b/c Area changes from increasing to decreasing @ $x = \sqrt{3}$

$\therefore$ Maximum Area $= 18(\sqrt{3}) - 2(\sqrt{3})^3 = 18\sqrt{3} - 6\sqrt{3} = \boxed{12\sqrt{3}}$

2)
$$f(x) = \begin{cases} 1 - 2\sin x, & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0 \end{cases}$$

a) Proving continuity @ $x = c$: (1. show $\lim_{x\to c} f(x)$ exists / 2. $f(c)$ exists / 3. $\lim_{x\to c} f(x) = f(c)$.)

1. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1-2\sin x = 1-2\sin(0) = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-4x} = e^{0} = 1$

since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ then $\lim_{x\to 0} f(x) = 1$, exists

2. $f(0) = 1-2\sin(0) = 1$

3. $\lim_{x\to 0} f(x) = 1 = f(0)$

$\therefore$ $f(x)$ is continuous @ $x=0$.

b) $f'(x) = \begin{cases} -2\cos x & \text{for } x \le 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$f'(x) = -3$ (have to test both pieces of $f(x)$).

for $x \le 0$: $-2\cos x = -3$
$\cos x = \frac{3}{2} = 1\frac{1}{2}$ but $-1 \le \cos x \le 1$. Thus, $\cos x$ can never $= 1\frac{1}{2}$ b/c the biggest it can get is 1.

$\therefore$ No soln to $-2\cos x = -3$

for $x > 0$: $-4e^{-4x} = -3$

$e^{-4x} = \frac{3}{4}$

$-4x = \ln\left(\frac{3}{4}\right)$

$x = \boxed{\dfrac{\ln\left(\frac{3}{4}\right)}{-4}}$

c) Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{1-(-1)}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0} 1-2\sin x\,dx + \int_0^1 e^{-4x}\,dx\right]$

$= \frac{1}{2}\left(x+2\cos x\Big|_{-1}^{0}\right) + \frac{1}{2}\left(\frac{e^{-4x}}{-4}\Big|_0^1\right)$

$= \frac{1}{2}\left[2-(-1+2\cos(-1))\right] + \frac{1}{2}\left(\frac{e^{-4}}{-4}+\frac{1}{4}\right)$

$$= \frac{1}{2}\left(3 - 2\cos(-1)\right) + \frac{1}{2}\left(-\frac{1}{4e^4} + \frac{1}{4}\right)$$

$$= \frac{3}{2} - \cos(-1) - \frac{1}{8e^4} + \frac{1}{8} = \boxed{\frac{13}{8} - \frac{1}{8e^4} - \cos(-1)}$$

3) $V = \pi r^2 h$, $r = 5$, $\frac{dV}{dt} = -5\pi\sqrt{h}$

a) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ b/c r is a coeff.

$$-5\pi\sqrt{h} = \pi(5)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

b) $h(0) = 17$

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$$

$$\int \frac{1}{\sqrt{h}}\,dh = \int -\frac{1}{5}\,dt$$

$$\int h^{-1/2}\,dh = -\frac{1}{5}t + C$$

$$2h^{1/2} = -\frac{1}{5}t + c$$

$$2\sqrt{17} = -\frac{1}{5}(0) + c \Rightarrow C = 2\sqrt{17}$$

$$\therefore\ 2\sqrt{h} = -\frac{1}{5}t + 2\sqrt{17}$$

solve for h

$$\sqrt{h} = -\frac{1}{10}t + \sqrt{17}$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

c) The coffee pot is empty $\rightarrow h = 0$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$$

$$-\frac{1}{10}t + \sqrt{17} = 0$$

$$t = 10\sqrt{17} \text{ seconds}$$

4) $\int g'(x) = g(x)$

a) $\int_0^3 g'(x) = g(3) - g(0)$

$g(3) = \underbrace{\int_0^3 g'(x)\,dx}_{}+ 5$

can be found by finding the area between $g'(x)$ and the x-axis.

Side work: finding $\int_0^3 g'(x)\,dx$ = Area from $x=0$ to $x=2$ + Area from $x=2$ to $x=3$

$= \frac{\pi r^2}{4} + \frac{1}{2}bh$

$= \frac{\pi(2)^2}{4} + \frac{1}{2}(1)(3) = \pi + \frac{3}{2}$

$g(3) = \int_0^3 g'(x)\,dx + 5$

$\Rightarrow g(3) = \pi + \frac{3}{2} + 5 = \boxed{\pi + \frac{13}{2}}$

$\int_0^{-2} g'(x)\,dx = g(-2) - g(0)$

$-\int_{-2}^{0} g'(x)\,dx = g(-2) - g(0)$

$-\frac{\pi r^2}{4} = g(-2) - 5 \implies g(-2) = \boxed{5 - \pi}$

b) pts. of inflection @ $x = 0, 2, 3$ b/c $g''(x)$ changes from + to − @ $x = 0, 3$ and $g''(x)$ changes from − to + @ $x = 2$.

c) $h(x) = g(x) - \frac{1}{2}x^2$

$h'(x) = g'(x) - x = 0$

$g'(x) = x$ or for the graph of $g'(x)$: $y = x$

$g'(x) = x$ @ $x = 3$ b/c $g'(3) = 3$

There is another pt. on the graph of $g'(x)$ where $y = x$.
It is somewhere between $x=1$ and $x=2$.

To find it: 1st have to derive the equation of the circle

general equation of a circle: $(x-h)^2+(y-k)^2=r^2$ with a center of (h,k) and a radius of r.

$\therefore$ Eqn of circle in graph of $g'(x)$ is

$x^2+y^2=4$

solving for y: $y=\sqrt{4-x^2}$

Remember, we are trying to figure out where $y=x$

$\therefore$ $\sqrt{4-x^2}=x$

$4-x^2=x^2$

$4=2x^2$

$x^2=2$

$x=\pm\sqrt{2}$ only $+\sqrt{2}$ is between 1 and 2

Critical pts. @ $x=\sqrt{2}$ and $x=3$

from graph

$0: h'(0)=g'(0)-0=2-0=2$

$2: h'(2)=g'(2)-2=0-2=-2$

$4: h'(4)=g'(4)-4=-$

Remember to plug into $h'(x)$ see side

$\therefore$ Relative maximum @ $x=\sqrt{2}$ b/c $h'(x)$ changes from $+$ to $-$ and $x=3$ is neither a relative max nor min b/c $h'(x)$ did not change signs.

5) a) See Review WS 3 for graph of slope field

b) $\frac{dy}{dx}=\frac{x+1}{y}$

$\frac{x+1}{y}=-1$

$x+1=-y$ All the points that satisfy $\frac{dy}{dx}=-1$ lie on the line $y = -x - 1$

c) $f(0) = -2$

$$\frac{dy}{dx} = \frac{x+1}{y}$$

$$\int y\,dy = \int x+1\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + x + C$$

$$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C$$

$$2 = C$$

$$\frac{y^2}{2} = \frac{x^2}{2} + x + 2$$

$$y^2 = x^2 + 2x + 4$$

$$y = \pm\sqrt{x^2+2x+4}$$

But since $f(0) = -2$ then $\boxed{y = -\sqrt{x^2+2x+4}}$