Motion

There are two different types of word problems associated with motion: Related Rates and the relationship between position, velocity and acceleration of a particle.

Related Rates: The idea behind these problems is very simple. In a typical problem, you'll be given an equation of relating two or more variables. These variables will change with respect to time, and you'll use derivatives to determine how the rates are related. (Hence the name: related rates.)

1. A circular pool of water is expanding at the rate of $16\pi \frac{in^2}{\sec}$. At what rate is the radius expanding when the radius is 4 inches?
2. A 25-foot long ladder is leaning against the wall and sliding toward the floor. If the foot of the ladder is sliding away from the base at a rate of 15 ft/sec, how fast is the top of the ladder sliding down the wall when the top of the ladder is 7 feet from the ground.
3. A spherical balloon is expanding at a rate of $60\pi \frac{in^3}{\sec}$. How fast is the surface area of the balloon expanding when the radius of the balloon is 4 in?
4. An underground conical tank, standing on its vertex, is being filled with water at the rate of $18\pi \frac{ft^3}{\min}$. If the tank has a height of 30 feet and a radius of 15 feet, how fast is the water level rising when the water is 12 feet deep?

Other examples:

1. A circle is increasing in area at the rate of $16\pi \frac{in^2}{\sec}$. How fast is the radius increasing when the radius is 2 in?

2. A rocket is rising vertically at a rate of 5,400 miles per hour. An observer on the ground is standing 20 miles from the rocket's launch point. How fast (in radians per second) is the angle of elevation between the ground and the observer's line of sight of the rocket increasing when the rocket is at an elevation of 40 miles? (Notice that the velocity is given in miles per hour and the answer asks for radians per second. In situations like this one, you have to be sure to convert the units properly or you'll get the wrong answer.)

Related Rates Examples

1)

$16\pi \frac{in^2}{sec}$ → Area → Area of a circle $A = \pi r^2$

sec → dt A and r change with time

deriv: $1\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

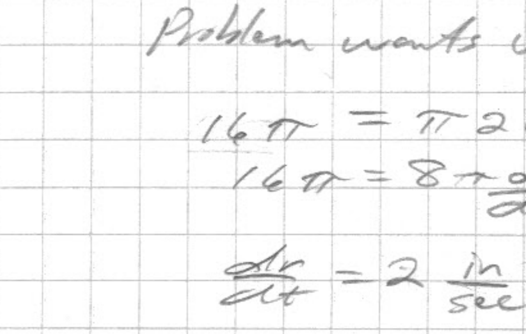
Problem wants us to solve for $\frac{dr}{dt}$.

$16\pi = \pi 2(4)\frac{dr}{dt}$

$16\pi = 8\pi \frac{dr}{dt}$

$\frac{dr}{dt} = 2 \frac{in}{sec}$ ∴ Radius is expanding @ a rate of 2 in/sec.

2) Popular AP Question – "Falling Ladder" Question



Pythagorean Th^{m}: $a^2+b^2=c^2$

a and b change with time

c does <u>not</u> change with time.

In the formula, $a^2+b^2=c^2$

c will be a constant and the deriv. of a constant is zero. Thus,

deriv: $2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$

problem wants us to solve for $\frac{da}{dt}$.

* $2(7)\frac{da}{dt} + 2b(15) = 0$

Need to solve for b in another formula, use the pythagorean Th^{m}.

$a^2+b^2=c^2$

$7^2+b^2=25^2$

$49+b^2=625$

$b=24$ Now, plug back into deriv. formula

$2(7)\frac{da}{dt} + 2(24)(15) = 0$

$14\frac{da}{dt} + 720 = 0$

$\frac{da}{dt} = -\frac{360}{7}$

Ladder is falling at a rate of $\frac{360}{7}$ ft/sec.

3) Problem asks for surface area of a sphere:

$A = 4\pi r^2$

A and r change with time

$\frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt}$

problem asks us to solve for $\frac{dA}{dt}$

✱ $\frac{dA}{dt} = 4\pi(2)(4)\frac{dr}{dt}$ ← need to solve for $\frac{dr}{dt}$ using a different formula

Problem gives you $60\pi \frac{in^3}{sec}$ → Volume → Volume of a sphere

$V = \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi r^3$

V and r change with time

$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$

$60\pi = \frac{4}{3}\pi(3)(16)\frac{dr}{dt}$

$60\pi = 64\pi\frac{dr}{dt}$

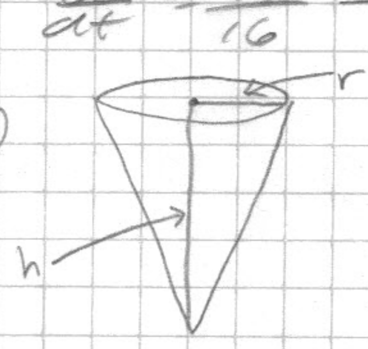
$\frac{dr}{dt} = \frac{15}{16}$ → plug back into deriv form.

$\frac{dA}{dt} = 4\pi(2)(4)\left(\frac{15}{16}\right)$

$\frac{dA}{dt} = 30\pi$

∴ Surface area is expanding at a rate of $30\pi \frac{in^2}{sec}$

4)



$18\pi \frac{ft^3}{min}$ → Volume → Volume of a cone

$V = \frac{1}{3}\pi r^2 h$ ← Do you foresee a problem with this formula.

V, r, and h change with time

✱ Before you take the derivative, Notice that the right side of the equal sign has 2 variables changing with time, so not only will it require a product rule but you will also have $\frac{dr}{dt}$ and $\frac{dh}{dt}$. The derivative formula would be:

$\frac{dV}{dt} = \frac{1}{3}\pi\left[2r\frac{dr}{dt}h + 1\frac{dh}{dt}r^2\right]$ ← way too many variables in this formula so don't use it.

instead, For these cone problems always solve for r or h using a proportion before you take the derivative.

How to solve for r or h in cone problems

The radius of the cone = 15 and the height of the cone = 30. Hence,

$$\frac{h}{r} = \frac{30}{15}$$

crossmultiply

$$15h = 30r$$

we want to find $\frac{dh}{dt}$ so we solve for r.

$$\frac{30r}{30} = \frac{15h}{30}$$

$$r = \frac{h}{2}$$ → substitute into formula before you take the deriv.

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)(h)$$

$$V = \frac{\pi}{12}h^3$$

Now, take the deriv. $\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$

solve for $\frac{dh}{dt}$

$$18\pi = \left(\frac{\pi}{12}\right)(3)(12)^2\frac{dh}{dt}$$

$$18\pi = 36\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = 2 \text{ ft/min}$$

∴ Water level is rising at a rate of $2\frac{ft}{min}$

Extra Examples:

1) Circle $16\pi \frac{in^2}{sec}$ → Area → Area of a circle

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

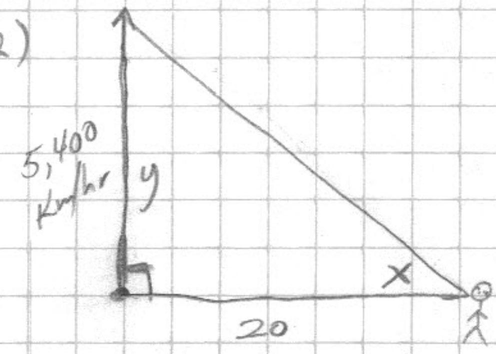
solve for $\frac{dr}{dt}$.

$$16\pi = \pi(2)(2)\frac{dr}{dt}$$

$$\frac{dr}{dt} = 4 \text{ in/sec}$$

radius is increasing at a rate of 4 in/sec

2)



$$\tan x = \frac{y}{20}$$

x and y change with time

$$(\sec^2 x)\frac{dx}{dt} = \frac{1}{20}\frac{dy}{dt}$$

solve for $\frac{dx}{dt}$

$\frac{dy}{dt} = 5400 \text{ Km/hr}$ but we want answer to be in radians/sec

so, change $\frac{dy}{dt} = \frac{5400 \text{ Km}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} = \frac{5400 \text{ Km}}{3600 \text{ sec}} = \frac{3}{2} \text{ Km/sec}$

need to solve for x by solving $\tan x = \frac{40}{20} = 2$

$x = \tan^{-1} 2 = 1.107$

plug back into deriv. formula

$$(\sec 1.107)^2 \frac{dx}{dt} = \frac{1}{20}\left(\frac{3}{2}\right)$$

$$5\frac{dx}{dt} = \frac{3}{40}$$

$$\frac{dx}{dt} = \frac{3}{200} \text{ radians/sec}$$