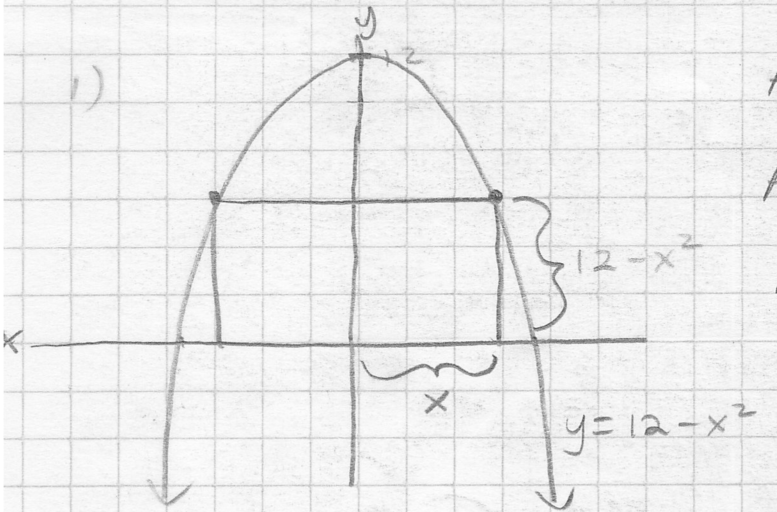
# Problem Set 10 WS Solutions

1)




Maximize area of rectangle

Area of rectangle $= l \cdot w$

$A_{(x)} = (2x)(12 - x^2)$

$A'_{(x)} = 2(12 - x^2) + (-2x)(2x)$

$A'(x) = 24 - 2x^2 - 4x^2 = 24 - 6x^2$

critical pts.

$24 - 6x^2 = 0$

$x^2 = 4$

$x = \pm 2$

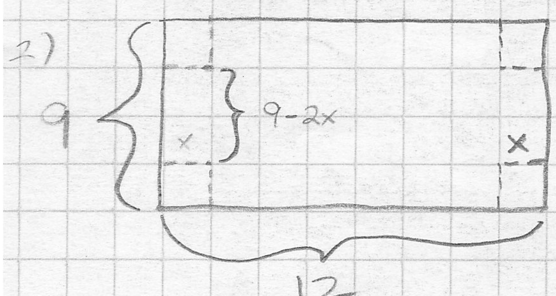
| $-3$ | $\boxed{-2}$ | $0$ | $\boxed{2}$ | $3$ |
|---|---|---|---|---|
| $-$ | | $+$ | | $-$ |

$\therefore$ local max @ $x = 2$

Maximum Area $= (2x)(12 - x^2) = 2(2)(12 - 2^2) = 32$

$\boxed{\text{32 square units}}$

2)




Maximize volume of box

$V = l \cdot w \cdot h$

$V = \underbrace{(12 - 2x)}_{l}\underbrace{(9 - 2x)}_{w}\underbrace{(x)}_{h}$

$V = (108 - 18x - 24x + 4x^2)\,x$

$V = 108x - 42x^2 + 4x^3$

$\frac{dV}{dx} = 108 - 84x + 12x^2$

critical points: $\frac{108 - 84x + 12x^2}{12} = 0$

$x^2 - 7x + 9 = 0$

$x = \frac{7 \pm \sqrt{49 - 36}}{2} = \frac{7 \pm \sqrt{13}}{2}$

Critical Pts @ $x = \frac{7 - \sqrt{13}}{2}$, $x = \frac{7 + \sqrt{13}}{2}$

$x = 1.697 \;,\; x = 5.303$

$x = 5.303$ doesn't make sense in the problem.
so, $x = 1.697$ is only critical pt.

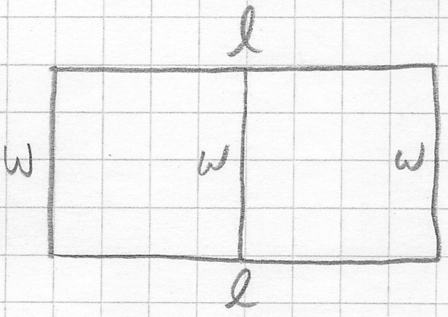
Second derivative test for local extrema

$$\frac{d^2V}{dx^2} = -84 + 24x$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=1.697} = -84 + 24(1.697) = (-) \Longrightarrow \text{local max @ } x = 1.697$$

$\boxed{\therefore \text{ Volume attains it's maximum @ } x = 1.697}$

3)




Minimize Perimeter

$P = 2l + 3w \longleftarrow$ P is defined by 2 variables. Need to turn into one variable.

The other info that was given $384\ m^2$ is the area of the outer rectangle

Area of a rect. $= l \cdot w$

$384 = l \cdot w$

solve for any variable : $l = \dfrac{384}{w}$ and substitute into Perimeter, P, equation

$$P = 2\left(\frac{384}{w}\right) + 3w = \frac{768}{w} + 3w \longleftarrow \text{just one variable now, } w.$$

$$\frac{dP}{dw} = 768(-1)w^{-2} + 3 = 3 - \frac{768}{w^2} = \frac{3w^2 - 768}{w^2}$$

critical points:

$3w^2 - 768 = 0$ $\qquad\qquad$ $w^2 = 0$

$w^2 = 256$ $\qquad\qquad$ $w = 0$

$w = \pm 16$

$w$ is width, width of $-16$ and $0$ makes no sense.

Only critical pt. that makes sense is $w=16$

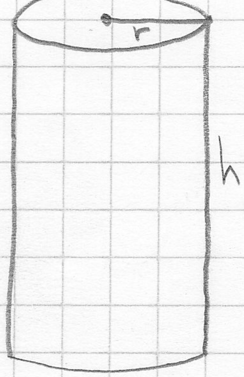
| 1 | $\boxed{16}$ | 20 |
|---|---|---|
| $-$ | | $+$ |

$\therefore$ local min @ $w=16$

$w = 16m \qquad l = \frac{384}{16} = 24m$

$\boxed{16m \times 24m}$

4)




Minimum material $\rightarrow$ minimize surface area

Surface area of a cylinder $= 2\pi r h + 2\pi r^2 \leftarrow$ defined by 2 variables

$V = 512 \text{ in}^3$

$V = \pi r^2 h$

$512 = \pi r^2 h \Rightarrow h = \frac{512}{\pi r^2}$

$$A = 2\pi r h + 2\pi r^2 = 2\cancel{\pi} r\left(\frac{512}{\cancel{\pi} r^2}\right) + 2\pi r^2 = \frac{1024}{r} + 2\pi r^2$$

$$\frac{dA}{dr} = \frac{-1024}{r^2} + 4\pi r = \frac{-1024}{r^2} + \frac{4\pi r^3}{r^2} = \frac{4\pi r^3 - 1024}{r^2}$$

critical pts.

$4\pi r^3 - 1024 = 0$

$r^3 = \frac{256}{\pi}$

$r = \sqrt[3]{\frac{256}{\pi}} = 4.335$

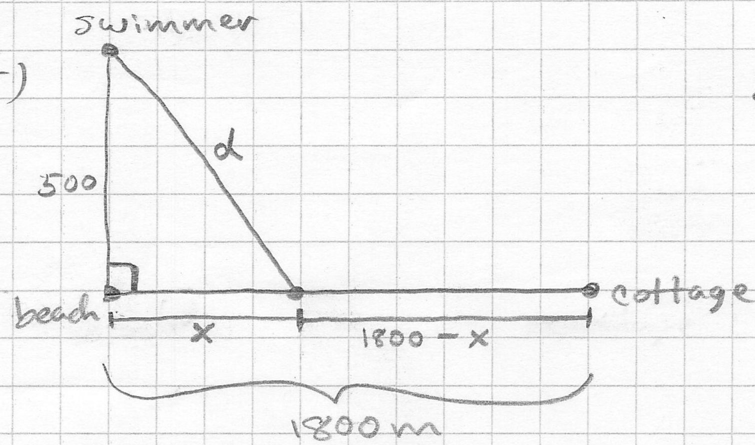
$r^2 = 0$

$r = 0 \leftarrow$ Doesn't make sense

| 1 | $\boxed{4.335}$ | 5 |
|---|---|---|
| $-$ | | $+$ |

$\therefore$ local min @ $r = 4.335$

$\boxed{\therefore \text{ Min material used when radius} = 4.335 \text{ in}}$

5) 


Shortest time → Minimize time

Distance = Rate · Time

$D = R \cdot T$

Use pythagorean formula to find $d$

$d^2 = x^2 + 500^2$

$d = \sqrt{x^2 + 250000}$

swims @ 4 m/s (R = 4 m/s)
walks @ 6 m/s (R = 6 m/s)

Swimming: $D = RT$

$d = 4t$

$\sqrt{x^2+250000} = 4t$

$t = \dfrac{\sqrt{x^2+250000}}{4}$

Walking: $D = RT$

$1800 - x = 6t$

$t = \dfrac{1800-x}{6}$

Total time is the <u>sum</u> of the 2 times above

$$\text{Total time} = \frac{\sqrt{x^2+250000}}{4} + \frac{1800-x}{6} = \frac{1}{4}\left(x^2+250000\right)^{1/2} + 300 - \frac{1}{6}x$$

$$\frac{dt}{dx} = \frac{1}{4}\left(\frac{1}{2}\right)\left(x^2+250000\right)^{-1/2}(2x) - \frac{1}{6} = \frac{x}{4\sqrt{x^2+250000}} - \frac{1}{6}$$

critical pts:

$$\frac{x}{4\sqrt{x^2+250000}} - \frac{1}{6} = 0$$

$$\frac{x}{4\sqrt{x^2+250000}} - \frac{1}{6} = \text{undef}$$

$4\sqrt{x^2+250000} = 0$

$x^2 + 250000 = 0$

$x^2 = -250000$

No soln.

Can solve using a common den. but...

$$\frac{x}{4\sqrt{x^2+250000}} = \frac{1}{6} \quad \text{cross multiply}$$

$$\left.\begin{aligned}(6x)^2 &= \left(4\sqrt{x^2+250000}\right)^2\\ 36x^2 &= 16(x^2+250000)\\ 36x^2 &= 16x^2 + 4000000\\ 20x^2 &= 4000000\end{aligned}\right\}\ \text{Easier, what do you think?}$$

$x^2 = 200000 \implies x = \pm 447.214$ (−447.214 Makes no sense)

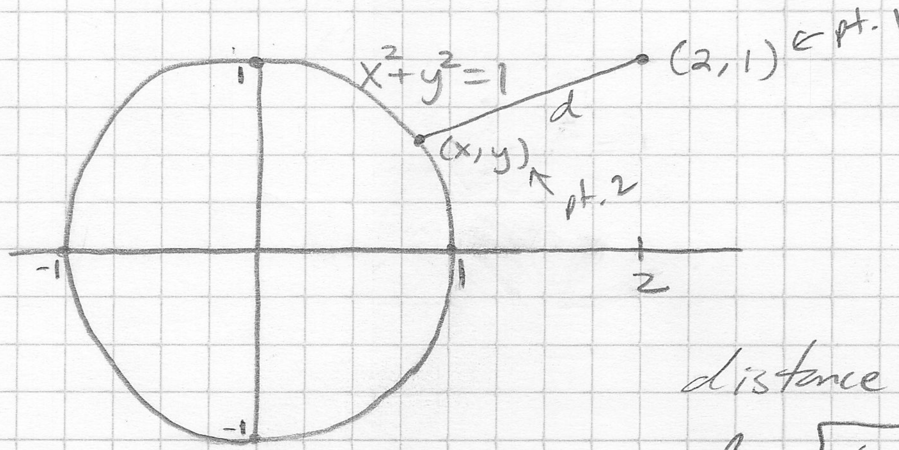
$\boxed{447.214}$

| 0 | | 500 |
|---|---|---|
| $-$ | | $+$ |

$\therefore$ local min @ $x = 447.214$

She should swim $1800 - x = 1800 - 447.214 = 1352.786\,m$ from the cottage

6)




Closest Point → Minimize distance between pts $(x,y)$ and $(2,1)$

distance formula:

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$d = \sqrt{(x-2)^2 + (y-1)^2}$$

$$d = \left[(x-2)^2 + (y-1)^2\right]^{1/2}$$

Problem: We work only with single variable calculus problems. Have to substitute for y. Find <u>another</u> equation with y in it.

$$x^2+y^2=1$$

solve for y: $y^2 = 1-x^2$

$$y = \sqrt{1-x^2}$$

Now, here's the ugly part, substitute into d

$$d = \left[(x-2)^2 + \left(\sqrt{1-x^2}-1\right)^2\right]^{1/2} = \left[x^2-4x+4+1-x^2-2\sqrt{1-x^2}+1\right]^{1/2}$$

$$d = \left[-4x - 2(1-x^2)^{1/2} + 6\right]^{1/2}$$

This is an <u>ugly</u> derivative. Check this out: square both sides and use implicit differentiation

$$d^2 = -4x - 2(1-x^2)^{1/2} + 6$$

$$2d\frac{dd}{dx} = -4 - 2\left(\tfrac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$$

$$2d\frac{dd}{dx} = -4 + 2x(1-x^2)^{-1/2}$$

divide every term by 2 : $\frac{2d\frac{dP}{dx}}{2} = \frac{-4}{2} + \frac{2x(1-x^2)^{-1/2}}{2}$

$d\frac{dP}{dx} = -2 + \frac{x}{\sqrt{1-x^2}} \Rightarrow \frac{dP}{dx} = \left(-2 + \frac{x}{\sqrt{1-x^2}}\right)\left(\frac{1}{d}\right)$

critical pts:

$\left(\frac{dP}{dx} = 0\right)$ $\left(-2 + \frac{x}{\sqrt{1-x^2}}\right)\left(\frac{1}{d}\right) = 0$

$\left(\frac{dP}{dx} = \text{undef}\right)$

Check other solns 1st

No point in trying to solve this one if you get the answer from $x = \pm\frac{2}{\sqrt{5}}$

$-2 + \frac{x}{\sqrt{1-x^2}} = 0$ or $\frac{1}{d} = 0$

No soln.

cross mult.

$\frac{x}{\sqrt{1-x^2}} = \frac{2}{1}$

$(x)^2 = (2\sqrt{1-x^2})^2$

$x^2 = 4(1-x^2)$

$x^2 = 4 - 4x^2$

$5x^2 = 4$

$x^2 = \frac{4}{5}$

$x = \pm\frac{2}{\sqrt{5}}$ (look at the picture, $x = -\frac{2}{\sqrt{5}}$ makes **No** sense).

| 0 | $\frac{2}{\sqrt{5}}$ | .9 |
|---|---|---|
| $-$ | | $+$ |

test with a calculator (Note: $\frac{1}{d}$ will always be positive)

$\therefore$ local min @ $x = \frac{2}{\sqrt{5}}$, $y = \sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1-\frac{4}{5}}$

$y = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$

$\therefore$ The closest point on the curve $x^2+y^2=1$ to $(2,1)$ is $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$