Limit Problems of $\frac{0}{0}$ WS

3 methods for evaluating limits in order:

1) Substitution
2) Factoring
3) Conjugating

1) $\lim\limits_{x\to 3} \dfrac{5x^2-8x-13}{x^2-5} = \dfrac{5(3)^2-8(3)-13}{3^2-5} = \dfrac{45-24-13}{9-5} = \dfrac{8}{4} = 2$

Think about: why does substitution work?

2) $\lim\limits_{x\to 2} \dfrac{3x^2-x-10}{x^2-4} = \dfrac{3(2)^2-2-10}{2^2-4} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$\lim\limits_{x\to 2} \dfrac{3x^2-x-10}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(3x+5)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \lim\limits_{x\to 2} \dfrac{3x+5}{x+2} = \dfrac{3(2)+5}{2+2} = \dfrac{11}{4}$

3) $\lim\limits_{x\to 3} \dfrac{x^4-81}{2x^2-5x-3} = \dfrac{3^4-81}{2(3)^2-5(3)-3} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$\lim\limits_{x\to 3} \dfrac{x^4-81}{2x^2-5x-3} = \lim\limits_{x\to 3} \dfrac{(x^2+9)(x^2-9)}{(2x+1)(x-3)} = \lim\limits_{x\to 3} \dfrac{(x^2+9)(x+3)\cancel{(x-3)}}{(2x+1)\cancel{(x-3)}}$

$= \lim\limits_{x\to 3} \dfrac{(x^2+9)(x+3)}{(2x+1)} = \dfrac{(3^2+9)(3+3)}{2(3)+1} = \boxed{\dfrac{108}{7}}$

4) $\lim\limits_{x\to -2} \dfrac{\frac{1}{x}+\frac{1}{2}}{x^3+8} = \dfrac{-\frac{1}{2}+\frac{1}{2}}{(-2)^3+8} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$\lim\limits_{x\to -2} \dfrac{\left(\frac{2}{2}\right)\frac{1}{x}+\frac{1}{2}\left(\frac{x}{x}\right)}{x^3+8} = \lim\limits_{x\to -2} \dfrac{\frac{2}{2x}+\frac{x}{2x}}{(x+2)(x^2-2x+4)} = \lim\limits_{x\to -2} \dfrac{\frac{(2+x)}{(2x)}}{(x+2)(x^2-2x+4)}$

$= \lim\limits_{x\to -2} \dfrac{\cancel{(2+x)}}{(2x)\cancel{(x+2)}(x^2-2x+4)} = \lim\limits_{x\to -2} \dfrac{1}{(2x)(x^2-2x+4)} = \dfrac{1}{2(-2)((-2)^2-2(-2)+4)}$

$= \boxed{\dfrac{1}{-48}}$

5) $\lim\limits_{x\to 4} \dfrac{3-\sqrt{x+5}}{x-4} = \dfrac{3-3}{4-4} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$\lim\limits_{x\to 4} \dfrac{(3-\sqrt{x+5})}{(x-4)} \dfrac{(3+\sqrt{x+5})}{(3+\sqrt{x+5})} = \lim\limits_{x\to 4} \dfrac{9-(x+5)}{(x-4)(3+\sqrt{x+5})} = \lim\limits_{x\to 4} \dfrac{4-x}{(x-4)(3+\sqrt{x+5})}$

$= \lim\limits_{x\to 4} \dfrac{-1(-4+x)}{(x-4)(3+\sqrt{x+5})} = \lim\limits_{x\to 4} \dfrac{-1\cancel{(x-4)}}{\cancel{(x-4)}(3+\sqrt{x+5})} = \dfrac{-1}{3+\sqrt{9}} = -\dfrac{1}{6}$

6) $\lim\limits_{x\to 27} \dfrac{x-27}{x^{1/3}-3} = \dfrac{27-27}{27^{1/3}-3} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$\lim\limits_{x\to 27} \dfrac{x-27}{x^{1/3}-3}$ Side-work: factor $x-27$ as a diff. of cubes

$x^{1/3}\,x^{1/3}\,x^{1/3}$ $3\,3\,3$

$= \lim\limits_{x\to 27} \dfrac{\cancel{(x^{1/3}-3)}(x^{2/3}+3x^{1/3}+9)}{\cancel{(x^{1/3}-3)}} = 27^{2/3}+3(27)^{1/3}+9 = 9+3(3)+9$

7) $\lim_{x \to 1} \dfrac{x^{1/3}-1}{x^{1/4}-1} = \dfrac{0}{0}$ indeterminate form

Think about: why did substitution fail?

$$\lim_{x \to 1} \frac{(x^{1/3}-1)}{(x^{1/4}-1)} \cdot \frac{(x^{2/3}+x^{1/3}+1)}{(x^{2/3}+x^{1/3}+1)} \cdot \frac{(x^{1/4}+1)(x^{1/2}+1)}{(x^{1/4}+1)(x^{1/2}+1)}$$

$$= \lim_{x \to 1} \frac{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)(x^{1/4}+1)(x^{1/2}+1)}{(x^{1/4}-1)(x^{1/4}+1)(x^{1/2}+1)(x^{2/3}+x^{1/3}+1)} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^{1/4}+1)(x^{1/2}+1)}{\cancel{(x-1)}(x^{2/3}+x^{1/3}+1)}$$

$$= \frac{(1^{1/4}+1)(1^{1/2}+1)}{1^{2/3}+1^{1/3}+1} = \frac{(2)(2)}{3} = \frac{4}{3}$$

8) $\lim_{x \to 0} \dfrac{\sin(5x)}{3x} = \dfrac{\sin(0)}{0} = \dfrac{0}{0}$ indeterminate form

Think about: why does substitution fail?

in order to do this problem you must know, that is, memorize that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$$\therefore \lim_{x \to 0} \frac{\sin(5x)}{3x} = \lim_{x \to 0} \frac{\sin(5x)}{3x} \cdot \frac{5}{5} = \lim_{x \to 0} \frac{5\sin(5x)}{3 \cdot 5x}$$

$$= \lim_{x \to 0} \frac{5}{3} \cdot \frac{\sin(5x)}{5x} = \lim_{x \to 0} \frac{5}{3} \cdot \lim_{x \to 0} \frac{\sin(5x)}{5x} = \frac{5}{3} \cdot 1 = \frac{5}{3}$$

9) $\lim_{x \to 0} \dfrac{\cos(2x)-1}{\cos x - 1} = \dfrac{\cos(0)-1}{\cos(0)-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ indeterminate form

Think about: You should know by now!

in order to do this problem recall from Trig that $\cos(2x) = 2\cos^2 x - 1$

$$\therefore \lim_{x\to 0} \frac{\cos(2x)-1}{\cos x - 1} = \lim_{x\to 0} \frac{2\cos^2 x - 1 - 1}{\cos x - 1} = \lim_{x\to 0} \frac{2\cos^2 x - 2}{\cos x - 1}$$

$$= \lim_{x\to 0} \frac{2(\cos^2 x - 1)}{\cos x - 1} = \lim_{x\to 0} \frac{2(\cos x + 1)\cancel{(\cos x - 1)}}{\cancel{\cos x - 1}} = 2(\cos(0)+1)$$

$$= 2(2) = 4$$

10) $\lim_{x\to 0} \frac{x^3 - 7x}{x^3} = \frac{0}{0}$ indeterminate form

$$\lim_{x\to 0} \frac{x^3-7x}{x^3} = \lim_{x\to 0} \frac{x(x^2-7)}{x^3} = \lim_{x\to 0} \frac{x^2-7}{x^2} = \frac{-7}{0}$$ Not an indeterminate form

Think about what a limit actually is.
What number is the numerator approaching?
what number is the denominator approaching?
∴ what number is the fraction approaching?

$$\lim_{x\to 0^-} \frac{x^2-7}{x^2} = \frac{-7}{0^+} = -\infty, \quad \lim_{x\to 0^+} \frac{x^2-7}{x^2} = \frac{-7}{0^+} = -\infty, \quad \therefore \lim_{x\to 0} \frac{x^2-7}{x^2} = -\infty$$

0^+ ← positive #

11) $\lim_{x\to 0} \frac{x^4+5x-3}{2-\sqrt{x^2+4}} = \frac{-3}{0}$ Not an indeterminate form

$$\lim_{x\to 0^-} \frac{x^4+5x-3}{2-\sqrt{x^2+4}} = \frac{-3}{0^-} = +\infty, \quad \lim_{x\to 0^+} \frac{x^4+5x-3}{2-\sqrt{x^2+4}} = \frac{-3}{0^-} = +\infty,$$

0^- ← negative #

$$\therefore \lim_{x\to 0} \frac{x^4+5x-3}{2-\sqrt{x^2+4}} = +\infty$$

12) $\lim_{x\to 1} \frac{x^3-1}{(x-1)^2} = \frac{0}{0}$ indeterminate form

$$\lim_{x\to 1} \frac{x^3-1}{(x-1)^2} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x-1)} = \lim_{x\to 1} \frac{1^2+1+1}{1-1} = \frac{3}{0}$$ Not an indet. form

However, $\lim_{x\to 1^-} \frac{x^2+x+1}{x-1} = \frac{3}{0^-} = -\infty, \quad \lim_{x\to 1^+} \frac{x^2+x+1}{x-1} = \frac{3}{0^+} = +\infty$

$$\therefore \lim_{x\to 1} \frac{x^2+x+1}{x-1} \text{ DNE}$$

13) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}} = \dfrac{\tan(\pi)}{0} = \dfrac{0}{0}$ indeterminate form

let $a = x - \frac{\pi}{2}$, thus $x = a + \frac{\pi}{2}$ and as $x \to \frac{\pi}{2}$

$a = \frac{\pi}{2} - \frac{\pi}{2} = 0$

So, $\lim\limits_{a \to 0} \dfrac{\tan\left[2\left(a + \frac{\pi}{2}\right)\right]}{a} = \lim\limits_{a \to 0} \dfrac{\tan(2a + \pi)}{a}$

Have to use trig identity: $\tan(A+B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

$$\lim_{a \to 0} \frac{\tan(2a+\pi)}{a} = \lim_{a \to 0} \frac{\tan(2a) + \tan(\pi)}{1 - \tan(2a)\tan(\pi)} \cdot \frac{1}{a} = \frac{\tan(2a) + 0}{1 - \tan(2a) \cdot 0} \cdot \frac{1}{a}$$

$$= \lim_{a \to 0} \frac{\tan(2a)}{1} \cdot \frac{1}{a} = \lim_{a \to 0} \frac{\sin(2a)}{\cos(2a)} \cdot \frac{1}{a} \frac{(2)}{(2)}$$

$$= \lim_{a \to 0} \frac{2}{\cos(2a)} \cdot \frac{\sin(2a)}{2a} = \frac{2}{\cos(0)} \cdot 1 = 2 \cdot 1 = 2$$

14) A. use a calculator

B.
a) 2
b) 1
c) DNE
d) 2
e) 2
f) 2
g) $-\infty$
h) 3
i) DNE
j) $\frac{1}{9}$
k) $-\frac{1}{9}$
l) 2.5

15) In order for $\lim\limits_{x \to 2} f(x)$ to exist then $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} a + bx = a + 2b$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} b - ax^2 = b - 4a$

Set each limit equal to $f(2)$

$\therefore\ a + 2b = 3 \Rightarrow a = 3 - 2b$

$b - 4a = 3$

$b - 4(3 - 2b) = 3$

$b - 12 + 8b = 3$

$9b = 15$

$b = \frac{5}{3}$

$a = 3 - 2\left(\frac{5}{3}\right)$

$a = \frac{9}{3} - \frac{10}{3} = -\frac{1}{3}$